On p-separability of subgroups of free metabelian groups

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Abstract

We prove that every free metabelian non–cyclic group has a finitely generated isolated subgroup which is not separable in the class of nilpotent groups.

As a corollary we prove that for every prime number p an arbitrary free metabelian non–cyclic group has a finitely generated p'–isolated subgroup which is not p–separable.

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Following A. I. Mal'cev [1] we say that a subgroup H of a group G is *separable in a class of groups* $\mathcal{K}$, if for each $g \in G \setminus H$ there is a homomorphism φ of G to some group from $\mathcal{K}$ such that $\varphi(g) \notin \varphi(H)$. If $\mathcal{K}$ is the class of all finite groups (resp. finite p–groups), the corresponding notion of $\mathcal{K}$-separability is called the *finite separability* (resp. *finite p–separability*). The problem of finite separability is closely related to the generalized word problem [1]. (The *generalized word problem* for H in G asks for an algorithm that decides whether or not the elements of G lie in H.)

Let p be a prime number. Recall that a subgroup H of G is called *p'–isolated*, if for every prime $q \neq p$ and for every $g \in G$ the condition $g^q \in H$ implies that $g \in H$. E. D. Loginova [2, § 3] proved that in each finitely generated nilpotent group every p'–isolated subgroup is p–separable.

D. I. Moldavanskii suggested that the latter fact is true in every free group.

Problem ([3, Problem 15.60]). *Is it true that any finitely generated p'–isolated subgroup of a free group is separable in the class of finite p–groups? It is easy to see that this is true for cyclic subgroups.*

The paper [4] gives the negative answer to Moldavanskii's problem. More precisely, it is proved there that in each non–abelian free group F there is an isolated (and therefore a p'–isolated) finitely generated subgroup which is not p–separable. In particular, the result by E. D. Loginova cannot be generalized to the class of absolute free groups.

In the present article we shall demonstrate that the result by E. D. Loginova cannot be generalized to the class of soluble groups, too. We prove the following theorem.

Theorem. *Every free metabelian non–cyclic group contains a finitely generated isolated subgroup which is not separable in the class of nilpotent groups.*

Since every finite p–group is nilpotent [5, p. 162], and every isolated subgroup is p'–isolated for each prime p, our theorem implies

Corollary. *Every free metabelian non–cyclic group contains a finitely generated p'–isolated subgroup which is not p–separable.*

§ 1. **The Magnus representation of free metabelian groups**

Let F denote a free group of rank n and F'' the second commutator subgroup of F. Then $\Phi = F/F''$ is a free metabelian group of rank n. W. Magnus [6, Chapter I, § 4] constructed a faithful 2×2 matrix representation of Φ. S. Bachmuth [7] studied the properties of the Magnus representation. For the reader's convenience we reproduce some properties of the Magnus representation.

Let Φ be freely generated by $x_1, x_2, \dots, x_n$. Let $s_1, s_2, \dots, s_n$, $t_1, t_2, \dots, t_n$ be commuting indeterminates. Then Φ has a faithful representation by 2×2 matrices defined via the correspondence

$$\mu : x_i \longrightarrow \begin{pmatrix} s_i & t_i \\ 0 & 1 \end{pmatrix}, \quad i = 1, 2, \dots, n.$$

Let $c = s_1^{j_1} s_2^{j_2} \dots s_n^{j_n}$, where $j_1, j_2, \dots, j_n$ are arbitrary integers. Suppose that

$$\gamma_1, \gamma_2, \dots, \gamma_n \in \mathbb{Z}[s_1^{\pm 1}, s_2^{\pm 1}, \dots, s_n^{\pm 1}].$$

Then the matrix

$$\begin{pmatrix} c & \sum\limits_{i=1}^{n} \gamma_i t_i \\ 0 & 1 \end{pmatrix}$$

is in the image of μ, or, equivalently, it is a product of matrices

$$\begin{pmatrix} s_i & t_i \\ 0 & 1 \end{pmatrix}^{\pm 1}$$

provided that the γ_i satisfy the identity

$$\sum_{j=1}^{n} \gamma_i (1 - s_i) = 1 - c$$

and vice versa (see [7, Lemma 1]).

§ 2. **Proof of the Theorem**

Let Φ be a free metabelian group with free generators

$$x, \ y, \ z_i \ (i \in I),$$

where I is some index set (possibly empty). Let H be a subgroup of Φ generated by the elements

$$a = x[y, x], \quad b = y, \quad z_j \ \ (j \in J),$$

where $[y, x] = y^{-1}x^{-1}yx$ and J is a subset of I.

We claim that H is a proper subgroup of Φ and, in particular, the element x is not in H. To do this let us consider a map τ from Φ to the symmetric group S_3 defined via

$$\tau : \begin{cases} x \longmapsto (12), \\ y \longmapsto (23), \\ z_i \longmapsto 1 \qquad \text{if} \quad i \in I. \end{cases}$$

Since S_3 is a metabelian group, τ is a homomorphism. It is easy to see that $\tau(a) = \tau(b) = (23)$, i. e., $\tau(H) = \langle (23) \rangle \simeq \mathbb{Z}_2$ and $\tau(x) \notin \tau(H)$. Hence $x \notin H$, as required.

Now let us prove that the subgroup H is not separable in the class of nilpotent groups. Consider the lower central series of Φ

$$\Phi = \gamma_1\Phi \geq \gamma_2\Phi \geq \gamma_3\Phi \geq \ldots,$$

where

$$\gamma_{i+1}\Phi = [\gamma_i\Phi, \Phi], \quad i = 1, 2, \ldots,$$

and the set of homomorphisms

$$\varphi_n : \Phi \longrightarrow \Phi/\gamma_n\Phi, \quad n = 1, 2, \ldots,$$

to the nilpotent metabelian groups. Note that the image of x under all these homomorphism is in the image of H. Indeed, it is easy to check that the subgroup generated by the elements

$$\varphi_n(a), \ \ \varphi_n(b), \ \ \varphi_n(z_j), \ \ j \in J,$$

and the subgroup generated by the elements

$$\varphi_n(x), \ \ \varphi_n(y), \ \ \varphi_n(z_j), \ \ j \in J,$$

are equal modulo the commutator subgroup $\Phi' = \gamma_2\Phi$.

Then (see [8, Theorem 31.2.5]) the group

$$\langle \varphi_n(a), \ \ \varphi_n(b), \ \ \varphi_n(z_j) \ \ (j \in J) \rangle$$

is equal to the group

$$\langle \varphi_n(x), \ \ \varphi_n(y), \ \ \varphi_n(z_j) \ \ (j \in J) \rangle.$$

Therefore $\varphi_n(x) \in \varphi_n(H)$ for all natural n. It then follows that H is not separable from element x in the class of free nilpotent groups and hence in the class of all nilpotent groups.

To complete the proof of the Theorem we must prove that H is isolated in Φ. For the sake of simplicity, we consider the case when the index set I is empty, i. e., Φ is generated by x and y. The proof in the general case is similar.

Now consider the matrices

$$X = \mu(x) = \begin{pmatrix} \alpha & e \\ 0 & 1 \end{pmatrix}, \quad Y = \mu(y) = \begin{pmatrix} \beta & f \\ 0 & 1 \end{pmatrix},$$

where $\{e, f\}$ is a base of a free module of rank 2 over the ring $R = \mathbb{Z}[\alpha^{\pm 1}, \beta^{\pm 1}]$. Lemma 1 from [7] implies that the matrix

$$\begin{pmatrix} \alpha^{i_1}\beta^{i_2} & \gamma_1 e + \gamma_2 f \\ 0 & 1 \end{pmatrix},$$

where $i_1, i_2 \in \mathbb{Z}$, $\gamma_1, \gamma_2 \in R$, is in $\mu(\Phi)$ if and only if

$$\gamma_1(1-\alpha) + \gamma_2(1-\beta) = 1 - \alpha^{i_1}\beta^{i_2}.$$

Note that there exist an epimorphism $\psi : \Phi \longrightarrow H$ that sends x to a and y to b. In this case we have the following diagram

$$\begin{array}{ccc} \Phi & \xrightarrow{\psi} & H \\ \mu \downarrow & & \downarrow \mu \\ M_2(Q) & \xrightarrow{\psi^*} & M_2(Q) \end{array}$$

where $Q = R[e, f]$. In order to define the map ψ^*, which makes this diagram to commutative we will find the images of generators of H under μ. Note that

$$X^{-1} = \begin{pmatrix} \alpha^{-1} & -\alpha^{-1}e \\ 0 & 1 \end{pmatrix}, \quad Y^{-1} = \begin{pmatrix} \beta^{-1} & -\beta^{-1}f \\ 0 & 1 \end{pmatrix},$$

which trivially implies that

$$\mu(a) = X[X, Y] = \begin{pmatrix} \alpha & (2-\beta^{-1})e + (1-\alpha)\beta^{-1}f \\ & \\ 0 & 1 \end{pmatrix},$$

$$\mu(b) = Y.$$

Write A for $\mu(a)$ and B for $\mu(b)$. We construct the homomorphism ψ^* via

$$\psi^*(X) = A, \quad \psi^*(Y) = B.$$

It is clear that the homomorphism λ of free module M defined as follows

$$\lambda : \begin{cases} e \longmapsto (2-\beta^{-1})e + (1-\alpha)\beta^{-1}f, \\ f \longmapsto f, \end{cases}$$

induces the homomorphism $\psi^* : M_2(Q) \longrightarrow M_2(Q)$. Then the map ψ^* takes a matrix

$$S = \begin{pmatrix} \alpha^{i_1}\beta^{i_2} & \gamma_1 e + \gamma_2 f \\ 0 & 1 \end{pmatrix} \in M_2(Q)$$

to the matrix

$$\psi^*(S) = \begin{pmatrix} \alpha^{i_1}\beta^{i_2} & \gamma_1(2-\beta^{-1})e + ((1-\alpha)\beta^{-1}\gamma_1 + \gamma_2)f \\ 0 & 1 \end{pmatrix}.$$

Now it is easy to see that this map is a homomorphism.

The matrix of the endomorphism λ in the base e, f is

$$\begin{pmatrix} 2-\beta^{-1} & (1-\alpha)\beta^{-1} \\ 0 & 1 \end{pmatrix}.$$

Its determinant is equal to $2-\beta^{-1} = (2\beta-1)/\beta$. Hence λ is not automorphism of M being considered as a module over $\mathbb{Z}[\alpha^{\pm 1}, \beta^{\pm 1}]$. The module M can be placed into the vector space $\widetilde{M}$ over field of rational fractions $\mathbb{Q}(\alpha, \beta)$ with the base e, f. Now λ induces an automorphism of $\widetilde{M}$; we shall denote the said induced map by $\widetilde{\lambda}$. Hence $\widetilde{\lambda} \in \operatorname{Aut}(\widetilde{M})$ does possess the inverse and

$$\widetilde{\lambda}^{-1} : \begin{cases} e \longmapsto \dfrac{\beta}{2\beta-1}e + \dfrac{\alpha-1}{2\beta-1}f, \\ \\ f \longmapsto f. \end{cases}$$

The map $\widetilde{\lambda}^{-1}$ induces a map ψ_1^* which sends the matrix

$$S = \begin{pmatrix} \alpha^{i_1}\beta^{i_2} & \gamma_1 e + \gamma_2 f \\ 0 & 1 \end{pmatrix} \in M_2(Q)$$

to the matrix

$$\psi_1^*(S) = \begin{pmatrix} \alpha^{i_1}\beta^{i_2} & \widetilde{\lambda}^{-1}(\gamma_1 e + \gamma_2 f) \\ 0 & 1 \end{pmatrix}.$$

However, it is not true in general that the latter matrix belongs to the ring $M_2(Q)$. The following simple lemma gives a criterion when it does belong to $M_2(Q)$.

Lemma 1. *The matrix $S \in \mu(\Phi)$ belongs to the subgroup $\mu(H)$ if and only if $\psi_1^*(S) \in \mu(\Phi)$.*

Let us return back to the proof of the Theorem. Consider the matrix S in $M_2(Q) \cap \mu(\Phi)$. By Bachmuth's lemma above, the following equality

$$\gamma_1(1-\alpha) + \gamma_2(1-\beta) = 1-c,$$

where $c = \alpha^{i_1}\beta^{i_2}$ is true. Assume that $S^m \in \mu(H)$ for some positive integer m.

The following lemma is obvious.

Lemma 2. *For every natural m and every matrix*

$$S = \begin{pmatrix} c & \gamma_1 e + \gamma_2 f \\ 0 & 1 \end{pmatrix} \in M_2(Q),$$

where $c = \alpha^{i_1}\beta^{i_2}$ for some $i_1, i_2 \in \mathbb{Z}$, the following formulas are true

$$S^m = \begin{pmatrix} c^m & (1+c+\ldots+c^{m-1})(\gamma_1 e + \gamma_2 f) \\ 0 & 1 \end{pmatrix},$$

$$S^{-m} = \begin{pmatrix} c^{-m} & -(c^{-1}+c^{-2}+\ldots+c^{-m})(\gamma_1 e + \gamma_2 f) \\ 0 & 1 \end{pmatrix}.$$

By Lemma 2 matrix S^m is equal to

$$S^m = \begin{pmatrix} c^m & (1+c+\ldots+c^{m-1})(\gamma_1 e + \gamma_2 f) \\ 0 & 1 \end{pmatrix}.$$

Then by Lemma 1 the matrix

$$\psi_1^*(S^m) = \begin{pmatrix} c^m & (1+c+\ldots+c^{m-1})\left(\frac{\gamma_1\beta}{2\beta-1}e + \left(\frac{\gamma_1(\alpha-1)}{2\beta-1} + \gamma_2\right) f\right) \\ 0 & 1 \end{pmatrix}$$

is in $M_2(Q)$. However, it is possible only in the case when $(1+c+\ldots+c^{m-1})\gamma_1\beta$ is a multiple of $2\beta-1$. Note that the polynomials β and $2\beta-1$ are relatively prime and hence the polynomial $1+c+\ldots+c^{m-1}$ is relatively prime with $2\beta-1$. Indeed, our first statement is evident while the second one follows from the fact that $c^{m-1} = (\alpha^{i_1}\beta^{i_2})^{m-1}$ and hence cannot be a multiple of 2. Therefore, γ_1 is a multiple of $2\beta-1$; but this means that $\psi_1^*(S) \in M_2(Q)$. We are going to show that $\psi_1^*(S) \in \mu(\Phi)$; then in view of Lemma 1 this will mean that $S \in \mu(H)$.

Consider the matrix

$$\psi_1^*(S) = \begin{pmatrix} c & \frac{\gamma_1\beta}{2\beta-1}e + \left(\frac{\gamma_1(\alpha-1)}{2\beta-1} + \gamma_2\right) f \\ 0 & 1 \end{pmatrix}.$$

By Bachmuth's lemma, this matrix is in $\mu(\Phi)$ provided that

$$\frac{\gamma_1\beta}{2\beta-1}(1-\alpha) + \left(\frac{\gamma_1(\alpha-1)}{2\beta-1} + \gamma_2\right)(1-\beta) = 1-c.$$

It is easy to check that this equality is equivalent to

$$\gamma_1(1-\alpha)(2\beta-1) + \gamma_2(1-\beta)(2\beta-1) = (1-c)(2\beta-1).$$

This implies that

$$\gamma_1(1-\alpha) + \gamma_2(1-\beta) = 1-c.$$

Since $S \in \mu(\Phi)$, it follows that the latter equality is true.

The Theorem is proven.

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REFERENCES

1. A. I. Mal’cev, On homomorphisms onto finite groups, Uchen. Zapiski Ivanovsk. ped. instituta, 18, N 5 (1958), 49–60 (also in “Selected papers”, Vol. 1, Algebra, 1976, 450–462) (Russian).

2. E. D. Loginova, Residual finiteness of the free product of two groups with commuting subgroups, Sib. Mat. Zh. 40, N 2 (1999), 395-407 (Russian).

3. The Kourovka Notebook (Unsolved problems in group theory), 15th ed., Institute of Mathematics SO RAN, Novosibirsk, 2002.

4. V. G. Bardakov, On D. I. Moldavanskii’s Question About p-Separable Subgroups of a Free Group, Siberian Mathematical Journal, 45, No 3 (2004), 416–419; see also arXiv : math.GR/0404292.

5. M. I. Kargapolov, Yu. I. Merzljakov, Fundamentals of the Theory of Groups, New York: Springer, 1979.

6. R. C. Lyndon, P. E. Schupp, Combinatorial group theory, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin-New York, 1977.

7. S. Bachmuth, Automorphisms of free metabelian groups, TAMS, 118, N 6 (1965), 93–104.

8. H. Neumann, Varieties of Groups, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 37, Springer–Verlag, Berlin–Heidelberg–New York, 1967.

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